

Perturbative Renormalization

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J} \cdot \text{s}, c = 2.998 \times 10^8 \text{ m/s}, 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}, \hbar c = 197 \text{ MeV fm}, k_B = 8.617 \times 10^{-5} \text{ eV/K}$$

$$1 \text{ GeV} = 10^3 \text{ MeV} = 10^6 \text{ keV} = 10^9 \text{ eV}, 1 \text{ m} = 100 \text{ cm} = 10^9 \text{ nm} = 10^{15} \text{ fm}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Contents

1	ϕ^3 theory	2
2	ϕ^4 theory	13
3	Complex ϕ^4 theory (working in progress)	22
4	pseudoscalar Yukawa theory (working in progress)	25
5	scalar QED (working in progress)	31
6	spinor QED (working in progress)	32
7	QCD (working in progress)	33
A	Useful formulas (working in progress)	34

1 ϕ^3 theory

We consider real scalar field ϕ . Bare ϕ^3 theory is given by

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{3!}g_0\phi_0^3. \quad (1)$$

For example, in the OS renormalization scheme, we require

$$\begin{aligned} \langle 0 | \phi(x) | 0 \rangle &= 0 \\ \langle k | \phi(x) | 0 \rangle &= e^{-ikx} \\ m &= m_{\text{phys}} \\ g &= g_{\text{phys}}, \end{aligned} \quad (2)$$

where m_{phys} is determined by the measurement of particle mass, and g_{phys} is determined by the measurement of cross section. Such renormalization conditions are satisfied by shifting the field values and rescaling the parameters

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + \frac{1}{3!}Z_g g\phi^3 + Y\phi. \quad (3)$$

This renormalized Lagrangian is decomposed into the free part and the interaction part

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\ \mathcal{L}_{\text{free}} &= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{\text{int}} &= \frac{1}{6}Z_g g\phi^3 + \mathcal{L}_{\text{CT}} \\ \mathcal{L}_{\text{CT}} &= -\frac{1}{2}A\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Bm^2\phi^2 + Y\phi. \end{aligned} \quad (4)$$

Here, $\mathcal{L}_{\text{CT}}$ is called a counter term, $A = Z_\phi - 1$, and $B = Z_m - 1$. The term $Y\phi$ in $\mathcal{L}_{\text{CT}}$ ensures

$$\langle 0 | \phi(x) | 0 \rangle = 0. \quad (5)$$

This means that the sum of all connected diagrams with single source is zero. No matter what this replacement subdiagram is, the sum of all connected diagrams with single source (tadpoles) is zero.

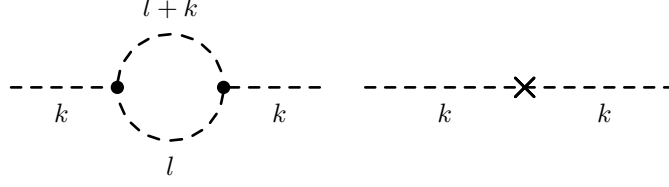


Figure 1: The 1 loop correction to the self energy

1 loop correction

Correction to the self energy

$$\begin{aligned}
i\Pi(k^2) &= \frac{1}{2} (ig)^2 \int \frac{d^d l}{(2\pi)^d} \left(\frac{-i}{l^2 - m^2 + i\epsilon} \right) \left(\frac{-i}{(l+k)^2 - m^2 + i\epsilon} \right) \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m^2) [(l+k)^2 - m^2]} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{1}{\left\{ (1-x)(l^2 - m^2) + x[(l+k)^2 - m^2] \right\}^2} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{1}{\left\{ (1-x)(l^2 - m^2) + x[l^2 + 2(k \cdot l) + (k^2 - m^2)] \right\}^2} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{1}{\left\{ l^2 + 2x(k \cdot l) + (1-x)(-m^2) + x(k^2 - m^2) \right\}^2} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{1}{\left\{ (l+xk)^2 - x^2 k^2 + (1-x)(-m^2) + x(k^2 - m^2) \right\}^2} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&= \frac{g^2}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{1}{\left\{ (l+xk)^2 + x(1-x)k^2 - m^2 \right\}^2} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(g^4) \\
&\equiv \frac{g^2}{2} \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \frac{1}{(q^2 + D)^2} - i (Ak^2 - Bm^2) + \mathcal{O}(g^4), \tag{6}
\end{aligned}$$

where

$$q \equiv l + xk \tag{7}$$

$$D \equiv x(1-x)k^2 - m^2. \tag{8}$$

Let's evaluate this loop integral

$$I(k^2) = \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2}. \quad (9)$$

We use Wick rotation to define a Euclidean d -dimensional vector $\bar{q}$ via

$$q^0 = \bar{q}_d, \quad q^j = i\bar{q}_j, \quad (10)$$

where

$$\bar{q}^2 = \bar{q}_1^2 + \bar{q}_2^2 + \cdots + \bar{q}_d^2. \quad (11)$$

To make g dimensionless, we will think of $d = 6$ case. But we perform dimensional regularization since $I(k^2)$ is divergent at $d = 6$. In general dimension d , the coupling g has the mass dimension of

$$[g] = \frac{\epsilon}{2} \quad (\epsilon \equiv 6 - d). \quad (12)$$

Therefore, we introduce scale parameter $\tilde{\mu}$ to separate the mass dimension from the coupling

$$g \rightarrow g\tilde{\mu}^{\epsilon/2} \quad (13)$$

Of course, any quantities must be independent of $\tilde{\mu}$ because $\tilde{\mu}$ is not the true parameter in $d = 6$ theory. If you apply this regularization to the loop integral, you get

$$\begin{aligned} I(k^2) &= i \int_0^1 dx \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \\ &= i \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} D^{-(2 - \frac{d}{2})} \\ &= \frac{i\Gamma(-1 + \frac{\epsilon}{2})}{(4\pi)^3} \int_0^1 dx D \left(\frac{4\pi}{D} \right)^{\epsilon/2} \quad (d = 6 - \epsilon), \end{aligned} \quad (14)$$

where we used

$$\int \frac{d^d \bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b - a - \frac{d}{2}) \Gamma(a + \frac{d}{2})}{(4\pi)^{d/2} \Gamma(b) \Gamma(d/2)} D^{-(b - a - \frac{d}{2})}. \quad (15)$$

Now using $g \rightarrow g\tilde{\mu}^{\epsilon/2}$ and $\alpha \equiv g^2 / (4\pi)^3$, we get

$$\begin{aligned} i\Pi(k^2) &= \frac{g^2}{2} I(k^2) - i(Ak^2 - Bm^2) + \mathcal{O}(g^4) \\ &= \frac{i}{2} g^2 \frac{\Gamma(-1 + \frac{\epsilon}{2})}{(4\pi)^3} \int_0^1 dx D \left(\frac{4\pi}{D} \right)^{\epsilon/2} - i(Ak^2 - Bm^2) + \mathcal{O}(g^4) \\ &\rightarrow \frac{i}{2} g^2 \tilde{\mu}^\epsilon \frac{\Gamma(-1 + \frac{\epsilon}{2})}{(4\pi)^3} \int_0^1 dx D \left(\frac{4\pi}{D} \right)^{\epsilon/2} - i(Ak^2 - Bm^2) + \mathcal{O}(g^4) \\ &= \frac{i}{2} \alpha \tilde{\mu}^\epsilon \Gamma(-1 + \frac{\epsilon}{2}) \int_0^1 dx D \left(\frac{4\pi}{D} \right)^{\epsilon/2} - i(Ak^2 - Bm^2) + \mathcal{O}(\alpha^2) \\ &= \frac{i}{2} \alpha \Gamma(-1 + \frac{\epsilon}{2}) \int_0^1 dx D \left(\frac{4\pi \tilde{\mu}^2}{D} \right)^{\epsilon/2} - i(Ak^2 - Bm^2) + \mathcal{O}(\alpha^2). \end{aligned} \quad (16)$$

Now, we take the limit of $\epsilon \rightarrow 0$. Using formula

$$\Gamma(-n + x) = \frac{(-1)^n}{n!} \left[\frac{1}{x} - \gamma + \sum_{k=1}^n \frac{1}{k} + \mathcal{O}(x) \right] \quad (n \in \mathbb{Z}^+, |x| \ll 1) \quad (17)$$

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2} \log A + \mathcal{O}(\epsilon^2), \quad (18)$$

we get

$$\begin{aligned}
\Gamma\left(-1 + \frac{\epsilon}{2}\right) \int_0^1 dx \, D \left(\frac{4\pi\tilde{\mu}^2}{D} \right)^{\epsilon/2} &= - \left[\frac{2}{\epsilon} - \gamma + 1 + \mathcal{O}\left(\frac{\epsilon}{2}\right) \right] \left[\int_0^1 dx \, D + \frac{\epsilon}{2} \int_0^1 dx \, D \log\left(\frac{4\pi\tilde{\mu}^2}{D} \right) + \mathcal{O}(\epsilon^2) \right] \\
&= - \left[\left(\frac{2}{\epsilon} + 1 \right) \left(\frac{1}{6} k^2 - m^2 \right) + \int_0^1 dx \, D \log\left(\frac{4\pi\tilde{\mu}^2}{\epsilon^\gamma D} \right) \right] \quad (\because -\gamma D = D \log e^{-\gamma}).
\end{aligned} \tag{19}$$

For the renormalization, we use $\overline{\text{MS}}$ scheme

$$\mu \equiv \sqrt{4\pi} e^{-\gamma/2} \tilde{\mu} \tag{20}$$

to get

$$\begin{aligned}
i\Pi(k^2) &= \frac{i}{2} \alpha \Gamma\left(-1 + \frac{\epsilon}{2}\right) \int_0^1 dx \, D \left(\frac{4\pi\tilde{\mu}^2}{D} \right)^{\epsilon/2} - i (Ak^2 - Bm^2) + \mathcal{O}(\alpha^2) \\
&= -\frac{i}{2} \alpha \left[\left(\frac{2}{\epsilon} + 1 \right) \left(\frac{1}{6} k^2 - m^2 \right) + \int_0^1 dx \, D \log\left(\frac{\mu^2}{D} \right) \right] - i (Ak^2 - Bm^2) + \mathcal{O}(\alpha^2) \\
&= \frac{i}{2} \alpha \int_0^1 dx \, D \log\left(\frac{D}{m^2} \right) \\
&\quad + \frac{i}{2} \alpha \left(\frac{1}{6} k^2 - m^2 \right) \log\left(\frac{\mu}{m} \right)^2 - i \alpha \left(\frac{1}{6} k^2 - m^2 \right) \frac{1}{\epsilon} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(\alpha^2) \\
&= \frac{i}{2} \alpha \int_0^1 dx \, D \log\left(\frac{D}{m^2} \right) \\
&\quad + i \alpha \left(\frac{1}{6} k^2 - m^2 \right) \log\left(\frac{\mu}{m} \right) - i \alpha \left(\frac{1}{6} k^2 - m^2 \right) \frac{1}{\epsilon} \\
&\quad - i (Ak^2 - Bm^2) + \mathcal{O}(\alpha^2) \\
&= \frac{i}{2} \alpha \int_0^1 dx \, D \log\left(\frac{D}{m^2} \right) \\
&\quad - i \left\{ \frac{\alpha}{6} \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) \right] + A \right\} k^2 \\
&\quad + i \left\{ \alpha \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) \right] + B \right\} m^2 + \mathcal{O}(\alpha^2).
\end{aligned} \tag{21}$$

Therefore, $i\Pi(k^2)$ becomes finite by setting

$$A = -\frac{\alpha}{6} \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) + \kappa_A \right] + \mathcal{O}(\alpha^2) \tag{22}$$

$$B = -\alpha \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) + \kappa_B \right] + \mathcal{O}(\alpha^2), \tag{23}$$

where κ_A, κ_B are the constants. This means that, at 1-loop level,

$$Z_\phi = 1 - \frac{\alpha}{6} \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) + \kappa_A \right] + \mathcal{O}(\alpha^2), \tag{24}$$

$$Z_m = 1 - \alpha \left[\frac{1}{\epsilon} - \log\left(\frac{\mu}{m} \right) + \kappa_B \right] + \mathcal{O}(\alpha^2). \tag{25}$$

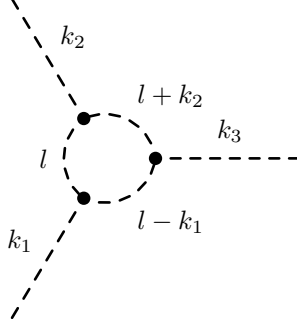


Figure 2: The 1 loop correction to the 3 point vertex function

Correction to the 3 point vertex function

$$\begin{aligned}
iV_3(k_1, k_2, k_3) &= iZ_g g + (ig)^3 \int \frac{d^d l}{(2\pi)^d} \left(\frac{-i}{l^2 - m^2 + i\epsilon} \right) \left(\frac{-i}{(l+k_2)^2 - m^2 + i\epsilon} \right) \left(\frac{-i}{(l-k_1)^2 - m^2 + i\epsilon} \right) + \mathcal{O}(g^5) \\
&= iZ_g g + g^3 \int \frac{d^d l}{(2\pi)^d} \left(\frac{1}{l^2 - m^2 + i\epsilon} \right) \left(\frac{1}{(l+k_2)^2 - m^2 + i\epsilon} \right) \left(\frac{1}{(l-k_1)^2 - m^2 + i\epsilon} \right) + \mathcal{O}(g^5) \\
&= iZ_g g + g^3 \int \frac{d^d l}{(2\pi)^d} \int dF_3 \frac{1}{[x_1(l-k_1)^2 + x_2(l+k_2)^2 + x_3 l^2 - m^2]^3} + \mathcal{O}(g^5), \tag{26}
\end{aligned}$$

where

$$\int dF_3 = (3-1)! \int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1). \tag{27}$$

Then

$$\begin{aligned}
&\int dF_3 \frac{1}{[x_1(l-k_1)^2 + x_2(l+k_2)^2 + x_3 l^2 - m^2]^3} \tag{28} \\
&= \int dF_3 \frac{1}{[l^2 - 2(x_1 k_1 - x_2 k_2) \cdot l + (x_1 k_1^2 + x_2 k_2^2 - m^2)]^3} \\
&= \int dF_3 \frac{1}{[(l - x_1 k_1 + x_2 k_2)^2 - (x_1 k_1 - x_2 k_2)^2 + (x_1 k_1^2 + x_2 k_2^2 - m^2)]^3} \\
&= \int dF_3 \frac{1}{\{q^2 + [x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1 x_2 k_1 k_2 - m^2]\}^3} \\
&\equiv \int dF_3 \frac{1}{[q^2 + D]^3} \tag{29}
\end{aligned}$$

where

$$q \equiv l - x_1 k_1 + x_2 k_2 \tag{30}$$

$$D \equiv x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1 x_2(k_1 \cdot k_2) - m^2. \tag{31}$$

Let's evaluate this loop integral

$$J(k_1, k_2, k_3) = \int dF_3 \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + D]^3} \tag{32}$$

Again, we perform Wick rotation and apply the dimensional regularization $d = 6 - \epsilon$. Separating the mass dimension from the coupling by a scale parameter $\tilde{\mu}$

$$g \rightarrow g \tilde{\mu}^{\epsilon/2}, \tag{33}$$

we get

$$\begin{aligned}
J(k_1, k_2, k_3) &= i \int dF_3 \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{[\bar{q}^2 + D]^3} \\
&= i \int dF_3 \frac{\Gamma(3 - \frac{d}{2})}{2(4\pi)^{d/2}} D^{-(3 - \frac{d}{2})} \\
&= i \int dF_3 \frac{\Gamma(\epsilon/2)}{2(4\pi)^{3 - \frac{\epsilon}{2}}} D^{-\epsilon/2}.
\end{aligned} \tag{34}$$

Then

$$\begin{aligned}
\frac{1}{g} V_3(k_1, k_2, k_3) &= Z_g + g^2 \tilde{\mu}^\epsilon \int dF_3 \frac{\Gamma(\epsilon/2)}{2(4\pi)^{3 - \frac{\epsilon}{2}}} D^{-\epsilon/2} + \mathcal{O}(g^3) \\
&= Z_g + \frac{\alpha}{2} \int dF_3 \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{4\pi \tilde{\mu}^2}{D}\right)^{\epsilon/2} + \mathcal{O}(\alpha^2) \quad (\alpha \equiv g^2 / (4\pi)^3) \\
&= Z_g + \frac{\alpha}{2} \left(\frac{2}{\epsilon} - \gamma\right) \int dF_3 \left[1 + \frac{\epsilon}{2} \log\left(\frac{4\pi \tilde{\mu}^2}{D}\right)\right] + \mathcal{O}(\alpha^2) \\
&= Z_g + \frac{\alpha}{2} \left[\frac{2}{\epsilon} + \int dF_3 \log\left(\frac{4\pi \tilde{\mu}^2}{e^\gamma D}\right)\right] + \mathcal{O}(\alpha^2) \\
&= Z_g + \frac{\alpha}{2} \left[\frac{2}{\epsilon} + \int dF_3 \log\left(\frac{\mu^2}{D}\right)\right] + \mathcal{O}(\alpha^2) \quad (\overline{\text{MS}} : \mu \equiv \sqrt{4\pi} e^{-\gamma/2} \tilde{\mu}).
\end{aligned} \tag{35}$$

If you write $Z_g = 1 + C$,

$$\frac{1}{g} V_3(k_1, k_2, k_3) = 1 + C + \alpha \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right)\right] - \frac{\alpha}{2} \int dF_3 \log\left(\frac{D}{m^2}\right) + \mathcal{O}(\alpha^2) \tag{36}$$

Therefore, $V_3(k_1, k_2, k_3)$ becomes finite by setting

$$C = -\alpha \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_C\right] + \mathcal{O}(\alpha^2), \tag{37}$$

where κ_C is a constant. This means that, at 1-loop level,

$$Z_g = 1 - \alpha \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_C\right] + \mathcal{O}(\alpha^2). \tag{38}$$

Renormalization group

From the perturbative calculation in $\overline{\text{MS}}$ scheme Eq. (24), (25) and (38), we have found

$$Z_\phi = 1 - \frac{\alpha}{6} \frac{1}{\epsilon} + \mathcal{O}(\alpha^2), \quad (39)$$

$$Z_m = 1 - \alpha \frac{1}{\epsilon} + \mathcal{O}(\alpha^2), \quad (40)$$

$$Z_g = 1 - \alpha \frac{1}{\epsilon} + \mathcal{O}(\alpha^2). \quad (41)$$

Here, we will see how the Lagrangian parameters m, g must change with the fake parameter μ from a formal point of view, and see how they extend to all orders of perturbation theory. Equations that tell us how the Lagrangian parameters vary with μ are collectively called the equations of the renormalization group.

Let us recall the Lagrangian of our theory, and write it in two different ways. In $d = 6 - \epsilon$ dimensions, we have the renormalized Lagrangian

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{3!} Z_g g \tilde{\mu}^{\epsilon/2} \phi^3 + Y \phi \quad (42)$$

and the bare Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{3!} g_0 \phi_0^3 + Y_0 \phi_0. \quad (43)$$

Comparing Eq (42) and Eq (43) gives us the relationships between the fields and parameters

$$\phi_0(x) = Z_\phi^{1/2} \phi, \quad (44)$$

$$m_0 = Z_\phi^{-1/2} Z_m^{1/2} m, \quad (45)$$

$$g_0 = Z_\phi^{-3/2} Z_g g \tilde{\mu}^{\epsilon/2}, \quad (46)$$

$$Y_0 = Z_\phi^{-1/2} Y. \quad (47)$$

Recall that, after using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of $\epsilon = 6 - d$. In the $\overline{\text{MS}}$ scheme, we choose the Z s to cancel off these powers of $1/\epsilon$ and nothing more. Therefore, the Z s can be written as

$$Z_\phi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n}, \quad (48)$$

$$Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{\epsilon^n}, \quad (49)$$

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\epsilon^n}, \quad (50)$$

where

$$a_1(\alpha) = -\frac{\alpha}{6} + \mathcal{O}(\alpha^2), \quad (51)$$

$$b_1(\alpha) = -\alpha + \mathcal{O}(\alpha^2), \quad (52)$$

$$c_1(\alpha) = -\alpha + \mathcal{O}(\alpha^2), \quad (53)$$

and that $a_n(\alpha)$, $b_n(\alpha)$, and $c_n(\alpha)$ are all at least $\mathcal{O}(\alpha^2)$ for $n \geq 2$. Now let us compute the beta function for α , the anomalous dimension of m and other useful things. This is the trick

bare fields and parameters must be independent of μ !

Let us start with g_0 . It is convenient to define

$$\alpha_0 \equiv \frac{g_0^2}{(4\pi)^3} = Z_\phi^{-3} Z_g^2 \tilde{\mu}^\epsilon \alpha \quad (54)$$

and also

$$G(\alpha, \epsilon) \equiv \log \left(Z_\phi^{-3} Z_g^2 \right). \quad (55)$$

From the general structure of Eq. (48), (49), and (50), we have

$$G(\alpha, \epsilon) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\epsilon^n}. \quad (56)$$

In particular,

$$\begin{aligned} G(\alpha, \epsilon) &\equiv \log \left(Z_\phi^{-3} Z_g^2 \right) \\ &= -3 \log Z_\phi + 2 \log Z_g \\ &= -3 \log \left(1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n} \right) + 2 \log \left(1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\epsilon^n} \right) \\ &\approx -3 \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n} + 2 \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\epsilon^n} \\ &= [-3a_1(\alpha) + 2c_1(\alpha) + \mathcal{O}(\alpha^2)] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \\ &= \left[-\frac{3}{2}\alpha + \mathcal{O}(\alpha^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}). \end{aligned} \quad (57)$$

Therefore

$$G_1(\alpha) = -\frac{3}{2}\alpha + \mathcal{O}(\alpha^2). \quad (58)$$

Now we have

$$\log \alpha_0 = G(\alpha, \epsilon) + \log \alpha + \epsilon \log \tilde{\mu}, \quad (59)$$

and if you differentiate this with respect to $\log \mu$ and require α_0 to be independent of it, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log \alpha_0 \quad (\mu^2 \equiv 4\pi e^{-\gamma} \tilde{\mu}^2) \\ &= \frac{\partial G(\alpha, \epsilon)}{\partial \alpha} \frac{d\alpha}{d \log \mu} + \frac{1}{\alpha} \frac{d\alpha}{d \log \mu} + \epsilon \end{aligned} \quad (60)$$

$$\Rightarrow 0 = \left(1 + \frac{\alpha G'_1(\alpha)}{\epsilon} + \frac{\alpha G'_2(\alpha)}{\epsilon^2} + \dots \right) \frac{d\alpha}{d \log \mu} + \epsilon \alpha \quad (61)$$

Here $\frac{d\alpha}{d \log \mu}$ is the rate at which α must change to compensate for a small change in $\log \mu$. If compensation is possible at all, this rate should be finite in the $\epsilon \rightarrow 0$ limit. Therefore, in a renormalizable theory, we should have

$$\frac{d\alpha}{d \log \mu} = -\epsilon \alpha + \beta(\alpha). \quad (62)$$

The result is

$$\begin{aligned} \beta(\alpha) &= \alpha^2 G'_1(\alpha) \\ &= -\frac{3}{2}\alpha^2 + \mathcal{O}(\alpha^3). \end{aligned} \quad (63)$$

Thus, as μ increases, $\alpha(\mu)$ decreases. A theory with this property is said to be *asymptotically free*. In this case, tree-level approximation (in the $\overline{\text{MS}}$ scheme with $\mu^2 \sim s$) becomes better and better at higher and higher energies.

Secondly, consider the invariance of m_0 . We begin by defining

$$\begin{aligned} M(\alpha, \epsilon) &\equiv \log\left(Z_\phi^{-1/2} Z_m^{1/2}\right) \\ &= \sum_{n=1}^{\infty} \frac{M_n(\alpha)}{\epsilon^n} \end{aligned} \quad (64)$$

In particular

$$\begin{aligned} M(\alpha, \epsilon) &\equiv \log\left(Z_\phi^{-1/2} Z_m^{1/2}\right) \\ &= -\frac{1}{2} \log Z_\phi + \frac{1}{2} \log Z_m \\ &\approx -\frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{\epsilon^n} \\ &= \left[-\frac{1}{2} a_1(\alpha) + \frac{1}{2} b_1(\alpha) + \mathcal{O}(\alpha^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \\ &= \left[-\frac{5}{12} \alpha + \mathcal{O}(\alpha^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \end{aligned} \quad (65)$$

Therefore

$$M_1(\alpha) = -\frac{5}{12} \alpha + \mathcal{O}(\alpha^2). \quad (66)$$

Now we have

$$\log m_0 = M(\alpha, \epsilon) + \log m. \quad (67)$$

By taking derivative of both sides with respect to $\log \mu$ and require m_0 to be independent of it, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log m_0 \quad (\mu^2 \equiv 4\pi e^{-\gamma} \tilde{\mu}^2) \\ &= \frac{\partial M(\alpha, \epsilon)}{\partial \alpha} \frac{d\alpha}{d \log \mu} + \frac{1}{m} \frac{dm}{d \log \mu} \end{aligned} \quad (68)$$

$$\begin{aligned} \Rightarrow \frac{1}{m} \frac{dm}{d \log \mu} &= (\epsilon \alpha - \beta(\alpha)) \left(\frac{M'_1(\alpha)}{\epsilon} + \frac{M'_2(\alpha)}{\epsilon^2} + \dots \right) \quad \left(\frac{d\alpha}{d \log \mu} = -\epsilon \alpha + \beta(\alpha) \right) \\ &= \alpha M'_1(\alpha) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \quad (69)$$

In a renormalized theory, $\frac{1}{m} \frac{dm}{d \log \mu}$ should be finite in the $\epsilon \rightarrow 0$ limit, and so $\mathcal{O}(1/\epsilon)$ terms must actually all be zero. Therefore, the *anomalous dimension of the mass*, defined via

$$\gamma_m(\alpha) \equiv \frac{1}{m} \frac{dm}{d \log \mu}, \quad (70)$$

is given by

$$\begin{aligned} \gamma_m(\alpha) &= \alpha M'_1(\alpha) \\ &= -\frac{5}{12} \alpha + \mathcal{O}(\alpha^2). \end{aligned} \quad (71)$$

Finally, let us take a look at the scaling rule of the correlation functions. In the $\overline{\text{MS}}$ scheme, the propagator in momentum space is given by

$$\tilde{\Delta}(k^2) \equiv \tilde{\Delta}(k^2; \mu, \alpha(\mu), m(\mu)) = i \int d^6x e^{-ikx} \langle 0 | T \phi(x) \phi(0) | 0 \rangle, \quad (72)$$

while the bare propagator is given by

$$\tilde{\Delta}_0(k^2) = i \int d^6x e^{-ikx} \langle 0 | T \phi_0(x) \phi_0(0) | 0 \rangle. \quad (73)$$

Here, the bare and renormalized propagators are related by

$$\tilde{\Delta}_0(k^2) = Z_\phi \tilde{\Delta}(k^2). \quad (74)$$

Now require that the bare propagator should be independent of μ . Taking the logarithm and differentiating with respect to $\log \mu$, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log \tilde{\Delta}_0(k^2) \\ &= \frac{d \log Z_\phi}{d \log \mu} + \frac{d}{d \log \mu} \log \tilde{\Delta}(k^2) \\ &= \frac{d \log Z_\phi}{d \log \mu} + \frac{1}{\tilde{\Delta}(k^2)} \left(\frac{\partial}{\partial \log \mu} + \frac{d\alpha}{d \log \mu} \frac{\partial}{\partial \alpha} + \frac{dm}{d \log \mu} \frac{\partial}{\partial m} \right) \tilde{\Delta}(k^2). \end{aligned} \quad (75)$$

We can write

$$\begin{aligned} \log Z_\phi &= \log \left(1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n} \right) \\ &\approx \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n}, \end{aligned} \quad (76)$$

then we have

$$\begin{aligned} \frac{d \log Z_\phi}{d \log \mu} &= \frac{\partial \log Z_\phi}{\partial \alpha} \frac{d\alpha}{d \log \mu} \\ &= \left(\frac{a'_1(\alpha)}{\epsilon} + \dots \right) (-\epsilon \alpha + \beta(\alpha)) \\ &= -\alpha a'_1(\alpha) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \quad (77)$$

Here the $\mathcal{O}(\frac{1}{\epsilon})$ term must all be zero because $\tilde{\Delta}(k^2)$ should vary smoothly with μ in the $\epsilon \rightarrow 0$ limit. We then define the *anomalous dimension of the field* by

$$\gamma_\phi(\alpha) \equiv \frac{1}{2} \frac{d \log Z_\phi}{d \log \mu}. \quad (78)$$

The result is

$$\begin{aligned} \gamma_\phi(\alpha) &= -\frac{1}{2} \alpha a'_1(\alpha) \\ &= \frac{1}{12} \alpha + \mathcal{O}(\alpha^2). \end{aligned} \quad (79)$$

Finally, in the $\epsilon \rightarrow 0$ limit, we get the *Callan-Symanzik equation* for the propagator

$$\left(\frac{\partial}{\partial \log \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_m(\alpha) m \frac{\partial}{\partial m} + 2\gamma_\phi(\alpha) \right) \tilde{\Delta}(k^2) = 0 \quad (80)$$

The Callan-Symanzik equation for the propagator

$$\left(\frac{\partial}{\partial \log \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_m(\alpha) m \frac{\partial}{\partial m} + 2\gamma_\phi(\alpha) \right) \tilde{\Delta}(k^2) = 0. \quad (81)$$

This is most interesting in the massless limit, and for a theory with a zero point of the beta function at a non-zero value of $\alpha = \alpha^*$ (fixed point)

$$\left(\frac{\partial}{\partial \log \mu} + 2\gamma_\phi(\alpha^*) \right) \tilde{\Delta}(k^2) = 0. \quad (82)$$

The solution is

$$\tilde{\Delta}(k^2) = \frac{C(\alpha^*)}{k^2} \left(\frac{\mu^2}{k^2} \right)^{-\gamma_\phi(\alpha^*)}, \quad (83)$$

where $C(\alpha^*)$ is an integration constant. Thus the naive scaling law

$$\tilde{\Delta}(k^2) \sim k^{-2} \quad (84)$$

is changed to

$$\tilde{\Delta}(k^2) \sim k^{-2[1-\gamma_\phi(\alpha^*)]}. \quad (85)$$

This has applications in the theory of critical phenomena.

2 ϕ^4 theory

We consider real scalar field ϕ with mass parameter $m^2 > 0$. Bare ϕ^4 theory is given by

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}\lambda_0\phi_0^4. \quad (86)$$

Renormalization conditions are satisfied by shifting the field values and rescaling the parameters

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - \frac{1}{4!}Z_\lambda\lambda\phi^4. \quad (87)$$

This renormalized Lagrangian is decomposed into the free part and the interaction part

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\ \mathcal{L}_{\text{free}} &= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{\text{int}} &= -\frac{1}{4!}Z_\lambda\lambda\phi^4 + \mathcal{L}_{\text{CT}} \\ \mathcal{L}_{\text{CT}} &= -\frac{1}{2}A\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Bm^2\phi^2. \end{aligned} \quad (88)$$

Here, $\mathcal{L}_{\text{CT}}$ is called a counter term, $A = Z_\phi - 1$, and $B = Z_m - 1$.

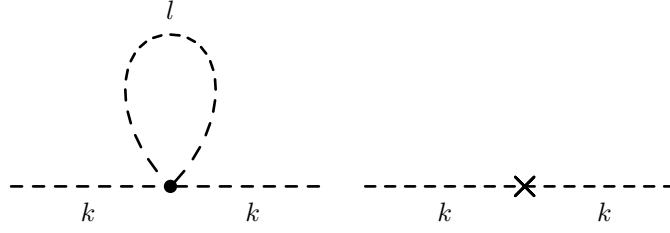


Figure 3: The 1 loop correction to the self energy

1 loop correction

Correction to the self energy

$$\begin{aligned}
i\Pi(k^2) &= \frac{1}{2}(-i\lambda) \int \frac{d^d l}{(2\pi)^d} \frac{-i}{l^2 - m^2 + i\epsilon} - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \\
&= -\frac{1}{2}\lambda \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2} - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \\
&\equiv -\frac{1}{2}\lambda \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + D} - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \quad (D \equiv -m^2)
\end{aligned} \tag{89}$$

Let's evaluate this loop integral

$$I = \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + D}. \tag{90}$$

We use Wick rotation to define a Euclidean d -dimensional vector $\bar{q}$ via

$$l^0 = \bar{q}_d, \quad l^j = i\bar{q}_j, \tag{91}$$

where

$$\bar{q}^2 = \bar{q}_1^2 + \bar{q}_2^2 + \cdots + \bar{q}_d^2. \tag{92}$$

To make λ dimensionless, we will think of $d = 4$ case. But we perform dimensional regularization since I is divergent at $d = 4$. In general dimension d , the coupling λ has the mass dimension of

$$[\lambda] = \epsilon \quad (\epsilon \equiv 4 - d). \tag{93}$$

Therefore, we introduce scale parameter $\tilde{\mu}$ to separate the mass dimension from the coupling

$$\lambda \rightarrow \lambda \tilde{\mu}^\epsilon \tag{94}$$

Of course, any quantities must be independent of $\tilde{\mu}$ because $\tilde{\mu}$ is not the true parameter in $d = 4$ theory. If you apply this regularization to the loop integral, you get

$$\begin{aligned}
I &= -i \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{\bar{q}^2 + D} \\
&= -i \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} D^{-(1 - \frac{d}{2})} \\
&= -\frac{i\Gamma(-1 + \frac{\epsilon}{2})}{(4\pi)^2} D \left(\frac{4\pi}{D}\right)^{\epsilon/2} \quad (d = 4 - \epsilon),
\end{aligned} \tag{95}$$

where we used

$$\int \frac{d^d \bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b - a - \frac{d}{2}) \Gamma(a + \frac{d}{2})}{(4\pi)^{d/2} \Gamma(b) \Gamma(d/2)} D^{-(b - a - \frac{d}{2})}. \tag{96}$$

Then we find

$$i\Pi(k^2) = i\frac{\lambda}{2(4\pi)^2}\tilde{\mu}^\epsilon\Gamma\left(-1+\frac{\epsilon}{2}\right)D\left(\frac{4\pi}{D}\right)^{\epsilon/2} - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2). \quad (D \equiv -m^2) \quad (97)$$

Now, we take the limit of $\epsilon \rightarrow 0$. Using formula

$$\Gamma(-n+x) = \frac{(-1)^n}{n!} \left[\frac{1}{x} - \gamma + \sum_{k=1}^n \frac{1}{k} + \mathcal{O}(x) \right] \quad (n \in \mathbb{Z}^+, |x| \ll 1) \quad (98)$$

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2} \log A + \mathcal{O}(\epsilon^2), \quad (99)$$

we get

$$\begin{aligned} i\Pi(k^2) &= -i\frac{\lambda}{2(4\pi)^2} \left[1 + \frac{\epsilon}{2} \log \tilde{\mu}^2 + \mathcal{O}\left(\frac{\epsilon}{2}\right) \right] \left[\frac{2}{\epsilon} - \gamma + 1 + \mathcal{O}\left(\frac{\epsilon}{2}\right) \right] D \left[1 + \frac{\epsilon}{2} \log\left(\frac{4\pi}{D}\right) + \mathcal{O}(\epsilon^2) \right] \\ &\quad - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \\ &= -i\frac{\lambda}{2(4\pi)^2} \left[1 + \frac{\epsilon}{2} \log \tilde{\mu}^2 + \mathcal{O}\left(\frac{\epsilon}{2}\right) \right] \left[\frac{2}{\epsilon} - \gamma + 1 + \mathcal{O}\left(\frac{\epsilon}{2}\right) \right] \left[1 + \frac{\epsilon}{2} \log\left(\frac{4\pi}{D}\right) + \mathcal{O}(\epsilon^2) \right] m^2 \\ &\quad - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \\ &= -i\frac{\lambda}{2(4\pi)^2} \left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right) + \mathcal{O}(\epsilon^2) \right] m^2 \\ &\quad - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2) \\ &= -i\frac{\lambda}{2(4\pi)^2} \left[\frac{2}{\epsilon} + 1 + \log\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon^2) \right] m^2 - i(Ak^2 - Bm^2) + \mathcal{O}(\lambda^2). \end{aligned} \quad (100)$$

Therefore, $i\Pi(k^2)$ becomes finite by setting

$$A = \mathcal{O}(\lambda^2), \quad (101)$$

$$B = \frac{\lambda}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_B \right] + \mathcal{O}(\lambda^2), \quad (102)$$

with fixed constant κ_B . This means that, at 1-loop level,

$$Z_\phi = 1 + \mathcal{O}(\lambda^2), \quad (103)$$

$$Z_m = 1 + \frac{\lambda}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_B \right] + \mathcal{O}(\lambda^2). \quad (104)$$

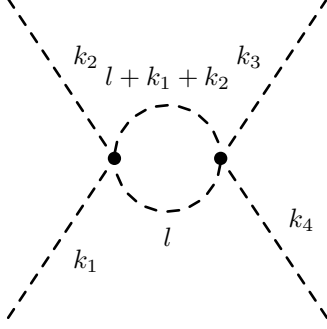


Figure 4: The 1 loop correction to the 4 point vertex function

Correction to the 4 point vertex function

$$\begin{aligned}
iV_4(k_1, k_2, k_3, k_4) &= -iZ_\lambda \lambda + \frac{1}{2} (-i\lambda)^2 \int \frac{d^d l}{(2\pi)^d} \left[\frac{-i}{l^2 - m^2 + i\epsilon} \right] \left[\frac{-i}{(l + k_1 + k_2)^2 - m^2 + i\epsilon} \right] \\
&\quad + (k_2 \leftrightarrow k_3) \\
&\quad + (k_2 \leftrightarrow k_4) \\
&\equiv -iZ_\lambda \lambda + \frac{1}{2} \lambda^2 [F(s) + F(t) + F(u)],
\end{aligned} \tag{105}$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_3)^2$, $u = (k_1 + k_4)^2$ and

$$F(k^2) \equiv \tilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[\frac{1}{l^2 - m^2} \right] \left[\frac{1}{(l + k)^2 - m^2} \right]. \tag{106}$$

Here, we performed the dimensional regularization $d = 4 - \epsilon$ with the scale parameter $\lambda \rightarrow \lambda \tilde{\mu}^\epsilon$. Then we get

$$\begin{aligned}
F(k^2) &\equiv \tilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[\frac{1}{l^2 - m^2} \right] \left[\frac{1}{(l + k)^2 - m^2} \right] \\
&= \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \left\{ \frac{1}{(1-x)(l^2 - m^2) + x[(l+k)^2 - m^2]} \right\}^2 \\
&= \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \left[\frac{1}{l^2 + 2x(l \cdot k) + xk^2 - m^2} \right]^2 \\
&= \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \left[\frac{1}{(l+xk)^2 + x(1-x)k^2 - m^2} \right]^2 \\
&\equiv \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2},
\end{aligned} \tag{107}$$

where

$$q \equiv l + xk \tag{108}$$

$$D \equiv x(1-x)k^2 - m^2 \tag{109}$$

Now we use Wick rotation to evaluate the loop integral

$$\begin{aligned}
F(k^2) &= \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} \\
&\rightarrow -i\tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \\
&= -i\tilde{\mu}^\epsilon \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} D^{-(2 - \frac{d}{2})} \\
&= -i\tilde{\mu}^\epsilon \frac{\Gamma(\epsilon/2)}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi}{D}\right)^{\epsilon/2} \quad (d = 4 - \epsilon). \tag{110}
\end{aligned}$$

Taking $\epsilon \rightarrow 0$ limit, we get

$$\begin{aligned}
F(k^2) &= i\tilde{\mu}^\epsilon \frac{\Gamma(\epsilon/2)}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi}{D}\right)^{\epsilon/2} \\
&= \frac{i}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \log \tilde{\mu}^2 + \mathcal{O}(\epsilon^2)\right] \left[\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right] \left[1 + \frac{\epsilon}{2} \int_0^1 dx \log\left(\frac{4\pi}{D}\right) + \mathcal{O}(\epsilon^2)\right] \\
&= \frac{i}{(4\pi)^2} \left[\frac{2}{\epsilon} + \int_0^1 dx \log\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right) + \mathcal{O}(\epsilon)\right] \\
&\equiv \frac{i}{(4\pi)^2} \left[\frac{2}{\epsilon} + \int_0^1 dx \log\left(\frac{\mu^2}{D}\right) + \mathcal{O}(\epsilon)\right] \quad (\overline{\text{MS}} : \mu = \sqrt{4\pi}e^{-\gamma}\tilde{\mu}). \tag{111}
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{1}{\lambda} V_4(k_1, k_2, k_3, k_4) &= -Z_\lambda + \frac{1}{2i} \lambda [F(s) + F(t) + F(u)] \\
&= -Z_\lambda + \frac{3\lambda}{16\pi^2} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right)\right] - \frac{1}{32\pi^2} \int_0^1 dx \log\left(\frac{D}{m^2}\right) + \mathcal{O}(\lambda^2) \tag{112}
\end{aligned}$$

Setting $Z_\lambda = 1 + C$, we see that

$$C = \frac{3\lambda}{16\pi^2} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_C\right] + \mathcal{O}(\lambda^2), \tag{113}$$

where κ_C is a fixed constant. This means that, at 1-loop level,

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2} \left[\frac{1}{\epsilon} + \log\left(\frac{\mu}{m}\right) + \kappa_C\right] + \mathcal{O}(\lambda^2). \tag{114}$$

Renormalization group

From the perturbative calculation in $\overline{\text{MS}}$ scheme Eq. (103), (104) and (114), we have found

$$Z_\phi = 1 + \mathcal{O}(\lambda^2), \quad (115)$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2} \frac{1}{\epsilon} + \mathcal{O}(\lambda^2), \quad (116)$$

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2} \frac{1}{\epsilon} + \mathcal{O}(\lambda^2). \quad (117)$$

Here, we will see how the Lagrangian parameters m, λ must change with the fake parameter μ from a formal point of view, and see how they extend to all orders of perturbation theory. Equations that tell us how the Lagrangian parameters vary with μ are collectively called the equations of the renormalization group.

Let us recall the Lagrangian of our theory, and write it in two different ways. In $d = 4 - \epsilon$ dimensions, we have the renormalized Lagrangian

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 - \frac{1}{4!} Z_\lambda \lambda \tilde{\mu}^\epsilon \phi^4, \quad (118)$$

and the bare Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!} \lambda_0 \phi_0^4. \quad (119)$$

Comparing Eq (42) and Eq (43) gives us the relationships between the fields and parameters

$$\phi_0(x) = Z_\phi^{1/2} \phi, \quad (120)$$

$$m_0 = Z_\phi^{-1/2} Z_m^{1/2} m, \quad (121)$$

$$\lambda_0 = Z_\phi^{-2} Z_\lambda \lambda \tilde{\mu}^\epsilon. \quad (122)$$

Recall that, after using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of $\epsilon = 4 - d$. In the $\overline{\text{MS}}$ scheme, we choose the Z s to cancel off these powers of $1/\epsilon$ and nothing more. Therefore, the Z s can be written as

$$Z_\phi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}, \quad (123)$$

$$Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n}, \quad (124)$$

$$Z_\lambda = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}, \quad (125)$$

where

$$a_1(\lambda) = 0 + \mathcal{O}(\lambda^2), \quad (126)$$

$$b_1(\lambda) = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \quad (127)$$

$$c_1(\lambda) = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \quad (128)$$

and that $a_n(\lambda)$, $b_n(\lambda)$, and $c_n(\lambda)$ are all at least $\mathcal{O}(\lambda^2)$ for $n \geq 2$. Now let us compute the beta function for λ , the anomalous dimension of m and other useful things. This is the trick

bare fields and parameters must be independent of μ !

Let us start with λ_0 . It is convenient to define

$$\lambda_0 = Z_\phi^{-2} Z_\lambda \tilde{\mu}^\epsilon \lambda \quad (129)$$

and also

$$L(\lambda, \epsilon) \equiv \log \left(Z_\phi^{-2} Z_\lambda \right). \quad (130)$$

From the general structure of Eq. (123), (124), and (125), we have

$$L(\lambda, \epsilon) = \sum_{n=1}^{\infty} \frac{L_n(\lambda)}{\epsilon^n}. \quad (131)$$

In particular,

$$\begin{aligned} L(\alpha, \epsilon) &\equiv \log \left(Z_\phi^{-2} Z_\lambda \right) \\ &= -2 \log Z_\phi + \log Z_\lambda \\ &= -2 \log \left(1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n} \right) + \log \left(1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n} \right) \\ &\approx -2 \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n} + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n} \\ &= [-2a_1(\lambda) + c_1(\lambda) + \mathcal{O}(\lambda^2)] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \\ &= \left[\frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}). \end{aligned} \quad (132)$$

Therefore

$$L_1(\lambda) = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2). \quad (133)$$

Now we have

$$\log \lambda_0 = L(\lambda, \epsilon) + \log \lambda + \epsilon \log \tilde{\mu}, \quad (134)$$

and if you differentiate this with respect to $\log \mu$ and require λ_0 to be independent of it, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log \lambda_0 \quad (\mu^2 \equiv 4\pi e^{-\gamma} \tilde{\mu}^2) \\ &= \frac{\partial L(\lambda, \epsilon)}{\partial \lambda} \frac{d\lambda}{d \log \mu} + \frac{1}{\lambda} \frac{d\lambda}{d \log \mu} + \epsilon \end{aligned} \quad (135)$$

$$\Rightarrow 0 = \left(1 + \frac{\lambda L'_1(\lambda)}{\epsilon} + \frac{\lambda L'_2(\lambda)}{\epsilon^2} + \dots \right) \frac{d\lambda}{d \log \mu} + \epsilon \lambda \quad (136)$$

Here $\frac{d\lambda}{d \log \mu}$ is the rate at which λ must change to compensate for a small change in $\log \mu$. If compensation is possible at all, this rate should be finite in the $\epsilon \rightarrow 0$ limit. Therefore, in a renormalizable theory, we should have

$$\frac{d\lambda}{d \log \mu} = -\epsilon \lambda + \beta(\lambda). \quad (137)$$

The result is

$$\begin{aligned} \beta(\lambda) &= \lambda^2 L'_1(\lambda) \\ &= \frac{3}{16\pi^2} \lambda^2 + \mathcal{O}(\lambda^3). \end{aligned} \quad (138)$$

The beta function is positive, which means that the theory becomes more and more strongly coupled at higher and higher energies.

Secondly, consider the invariance of m_0 . We begin by defining

$$\begin{aligned} M(\lambda, \epsilon) &\equiv \log\left(Z_\phi^{-1/2} Z_m^{1/2}\right) \\ &= \sum_{n=1}^{\infty} \frac{M_n(\lambda)}{\epsilon^n} \end{aligned} \quad (139)$$

In particular

$$\begin{aligned} M(\lambda, \epsilon) &\equiv \log\left(Z_\phi^{-1/2} Z_m^{1/2}\right) \\ &= -\frac{1}{2} \log Z_\phi + \frac{1}{2} \log Z_m \\ &\approx -\frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n} \\ &= \left[-\frac{1}{2} a_1(\lambda) + \frac{1}{2} b_1(\lambda) + \mathcal{O}(\lambda^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \\ &= \left[\frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2) \right] \frac{1}{\epsilon} + \mathcal{O}(\epsilon^{-2}) \end{aligned} \quad (140)$$

Therefore

$$M_1(\lambda) = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2). \quad (141)$$

Now we have

$$\log m_0 = M(\lambda, \epsilon) + \log m. \quad (142)$$

By taking derivative of both sides with respect to $\log \mu$ and require m_0 to be independent of it, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log m_0 \quad (\mu^2 \equiv 4\pi e^{-\gamma} \tilde{\mu}^2) \\ &= \frac{\partial M(\lambda, \epsilon)}{\partial \lambda} \frac{d\lambda}{d \log \mu} + \frac{1}{m} \frac{dm}{d \log \mu} \end{aligned} \quad (143)$$

$$\begin{aligned} \Rightarrow \frac{1}{m} \frac{dm}{d \log \mu} &= (\epsilon\lambda - \beta(\lambda)) \left(\frac{M'_1(\lambda)}{\epsilon} + \frac{M'_2(\lambda)}{\epsilon^2} + \dots \right) \quad \left(\frac{d\lambda}{d \log \mu} = -\epsilon\lambda + \beta(\lambda) \right) \\ &= \lambda M'_1(\lambda) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \quad (144)$$

In a renormalized theory, $\frac{1}{m} \frac{dm}{d \log \mu}$ should be finite in the $\epsilon \rightarrow 0$ limit, and so $\mathcal{O}(1/\epsilon)$ terms must actually all be zero. Therefore, the *anomalous dimension of the mass*, defined via

$$\gamma_m(\lambda) \equiv \frac{1}{m} \frac{dm}{d \log \mu}, \quad (145)$$

is given by

$$\begin{aligned} \gamma_m(\lambda) &= \lambda M'_1(\lambda) \\ &= \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2). \end{aligned} \quad (146)$$

Finally, let us take a look at the scaling rule of the correlation functions. In the $\overline{\text{MS}}$ scheme, the propagator in momentum space is given by

$$\tilde{\Delta}(k^2) \equiv \tilde{\Delta}(k^2; \mu, \lambda(\mu), m(\mu)) = i \int d^6x e^{-ikx} \langle 0 | T \phi(x) \phi(0) | 0 \rangle, \quad (147)$$

while the bare propagator is given by

$$\tilde{\Delta}_0(k^2) = i \int d^6x e^{-ikx} \langle 0 | T \phi_0(x) \phi_0(0) | 0 \rangle. \quad (148)$$

Here, the bare and renormalized propagators are related by

$$\tilde{\Delta}_0(k^2) = Z_\phi \tilde{\Delta}(k^2). \quad (149)$$

Now require that the bare propagator should be independent of μ . Taking the logarithm and differentiating with respect to $\log \mu$, we get

$$\begin{aligned} 0 &= \frac{d}{d \log \mu} \log \tilde{\Delta}_0(k^2) \\ &= \frac{d \log Z_\phi}{d \log \mu} + \frac{d}{d \log \mu} \log \tilde{\Delta}(k^2) \\ &= \frac{d \log Z_\phi}{d \log \mu} + \frac{1}{\tilde{\Delta}(k^2)} \left(\frac{\partial}{\partial \log \mu} + \frac{d\lambda}{d \log \mu} \frac{\partial}{\partial \lambda} + \frac{dm}{d \log \mu} \frac{\partial}{\partial m} \right) \tilde{\Delta}(k^2). \end{aligned} \quad (150)$$

We can write

$$\begin{aligned} \log Z_\phi &= \log \left(1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n} \right) \\ &\approx \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}, \end{aligned} \quad (151)$$

then we have

$$\begin{aligned} \frac{d \log Z_\phi}{d \log \mu} &= \frac{\partial \log Z_\phi}{\partial \lambda} \frac{d\lambda}{d \log \mu} \\ &= \left(\frac{a'_1(\lambda)}{\epsilon} + \dots \right) (-\epsilon \lambda + \beta(\lambda)) \\ &= -\lambda a'_1(\lambda) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \quad (152)$$

Here the $\mathcal{O}(\frac{1}{\epsilon})$ term must all be zero because $\tilde{\Delta}(k^2)$ should vary smoothly with μ in the $\epsilon \rightarrow 0$ limit. We then define the *anomalous dimension of the field* by

$$\gamma_\phi(\lambda) \equiv \frac{1}{2} \frac{d \log Z_\phi}{d \log \mu}. \quad (153)$$

The result is

$$\begin{aligned} \gamma_\phi(\lambda) &= -\frac{1}{2} \lambda a'_1(\lambda) \\ &= 0 + \mathcal{O}(\lambda^2). \end{aligned} \quad (154)$$

Finally, in the $\epsilon \rightarrow 0$ limit, we get the *Callan-Symanzik equation* for the propagator

$$\left(\frac{\partial}{\partial \log \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m} + 2\gamma_\phi(\lambda) \right) \tilde{\Delta}(k^2) = 0 \quad (155)$$

3 Complex ϕ^4 theory (working in progress)

We consider real scalar field ϕ with mass parameter $m^2 > 0$. Bare ϕ^4 theory is given by

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\Phi_0^\dagger\partial_\mu\Phi_0 - \frac{1}{2}m_0^2|\Phi_0|^2 - \frac{1}{4!}\lambda_0|\Phi_0|^4. \quad (156)$$

Renormalization conditions are satisfied by shifting the field values and rescaling the parameters

$$\mathcal{L} = -\frac{1}{2}Z_\Phi\partial^\mu\Phi^\dagger\partial_\mu\Phi - \frac{1}{2}Z_m m^2|\Phi|^2 - \frac{1}{4!}Z_\lambda\lambda|\Phi|^4. \quad (157)$$

This renormalized Lagrangian is decomposed into the free part and the interaction part

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\ \mathcal{L}_{\text{free}} &= -\frac{1}{2}\partial^\mu\Phi^\dagger\partial_\mu\Phi - \frac{1}{2}m^2|\Phi|^2 \\ \mathcal{L}_{\text{int}} &= -\frac{1}{4!}Z_\lambda\lambda|\Phi|^4 + \mathcal{L}_{\text{CT}} \\ \mathcal{L}_{\text{CT}} &= -\frac{1}{2}A\partial^\mu\Phi^\dagger\partial_\mu\Phi - \frac{1}{2}Bm^2|\Phi|^2. \end{aligned} \quad (158)$$

Here, $\mathcal{L}_{\text{CT}}$ is called a counter term, $A = Z_\Phi - 1$, and $B = Z_m - 1$.

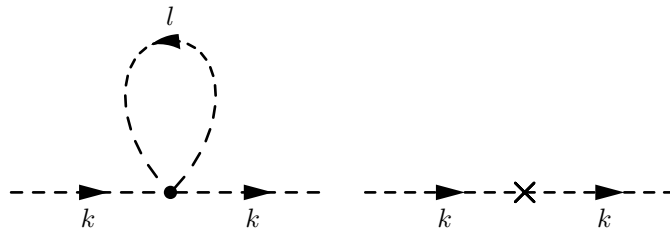


Figure 5: The 1 loop correction to the ϕ self energy

1 loop correction

Correction to the self energy

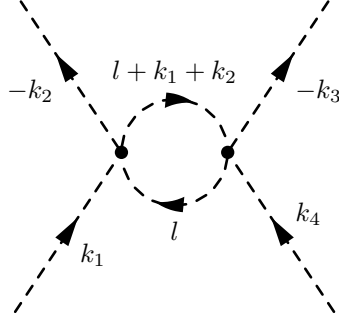


Figure 6: The 1 loop correction to the 4 point vertex function

Correction to the ϕ_2 4 point vertex function

4 pseudoscalar Yukawa theory (working in progress)

We consider the Yukawa theory of a spinor field and a real pseudoscalar field ϕ with mass parameter $m^2 > 0$. Bare ϕ^4 theory is given by

$$\mathcal{L} = i\bar{\Psi}_0 \not{\partial} \Psi_0 - m_0 \bar{\Psi}_0 \Psi_0 - \frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} M_0^2 \phi_0^2 + i g_0 \phi_0 \bar{\Psi}_0 \gamma_5 \Psi_0 - \frac{1}{4!} \lambda_0 \phi_0^4. \quad (159)$$

Renormalization conditions are satisfied by shifting the field values and rescaling the parameters

$$\mathcal{L} = \mathcal{L} = i Z_\Psi \bar{\Psi} \not{\partial} \Psi - Z_m m \bar{\Psi} \Psi - \frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + i Z_g g \phi \bar{\Psi} \gamma_5 \Psi - \frac{1}{4!} Z_\lambda \lambda \phi^4. \quad (160)$$

This renormalized Lagrangian is decomposed into the free part and the interaction part

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\ \mathcal{L}_{\text{free}} &= i\bar{\Psi} \not{\partial} \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} M^2 \phi^2 \\ \mathcal{L}_{\text{int}} &= i Z_g g \phi \bar{\Psi} \gamma_5 \Psi - \frac{1}{4!} Z_\lambda \lambda \phi^4 + \mathcal{L}_{\text{CT}} \\ \mathcal{L}_{\text{CT}} &= i A \bar{\Psi} \not{\partial} \Psi - B m \bar{\Psi} \Psi - \frac{1}{2} C \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} D M^2 \phi^2. \end{aligned} \quad (161)$$

Here, $\mathcal{L}_{\text{CT}}$ is called a counter term, $A = Z_\Psi - 1$, $B = Z_m - 1$, $C = Z_\phi - 1$, and $D = Z_M - 1$.

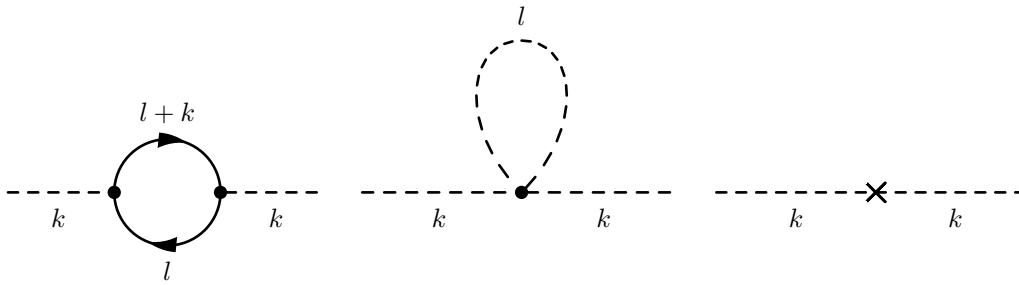


Figure 7: The 1 loop correction to the pseudoscalar self energy

1 loop correction

Correction to the pseudoscalar self energy

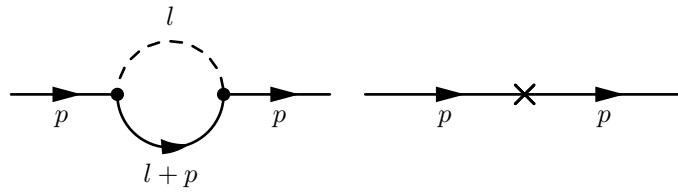


Figure 8: The 1 loop correction to the fermion self energy

Correction to the fermion self energy

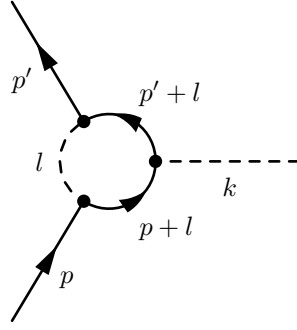


Figure 9: The 1 loop correction to the pseudo Yukawa vertex function

Correction to the pseudo Yukawa vertex function

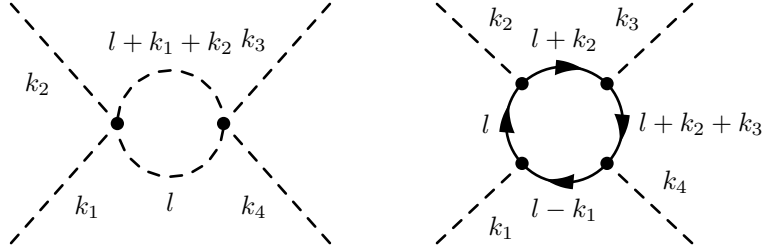


Figure 10: The 1 loop correction to the 4 point vertex function

Correction to the 4 point vertex function

Renormalization group

5 scalar QED (working in progress)

1 loop correction

Renormalization group

6 spinor QED (working in progress)

1 loop correction

Renormalization group

7 QCD (working in progress)

1 loop correction

Renormalization group

A Useful formulas (working in progress)